

Tues. 3 February 2014

Roots & Weights & SU(3)

Given a Lie group, the largest set of commuting, Hermitian generators forms a Cartan Subalgebra

In a particular representation D, there will be m generators, the Cartan generators, satisfying the rank of Cartan algebra

H_i^\dagger = H_i [H_i, H_j] = 0

The form a linear space

Tr(H_i H_j) = \lambda \delta_{ij}

In the diagonal representation for H_i,

H_i |mu, x, D> = mu_i |mu, x, D>

mu_i = weight

mu_i in R (Hermitian generator)

x = other labels needed to specify state

D = representation

m-component vector of mu_i = weight vector

Adjoint Representation

$$[T_a]_{bc} = -i f_{abc}$$

⇒ States of adjoint representation correspond to generators
(same index labels generators & elements of matrix)

$|X_a\rangle$ = state of arbitrary generator

Linear combinations of states correspond to linear combos of generators

$$\alpha |X_a\rangle + \beta |X_b\rangle = |\alpha X_a + \beta X_b\rangle$$

A convenient scalar product

$$\langle X_a | X_b \rangle = \lambda^{-1} \text{tr}(X_a^\dagger X_b) \quad (\alpha, \beta \in \mathbb{C})$$

What is action of generator on state?

$$\begin{aligned} X_a |X_b\rangle &= 1 * X_a |X_b\rangle \\ &= \sum_c |X_c\rangle \langle X_c | X_a |X_b\rangle \quad (\text{dry explicit sum}) \end{aligned}$$

$$= \sum_c |X_c\rangle [T_a]_{cb}$$

$$= -i f_{acb} |X_c\rangle$$

$$= +i f_{abc} |X_c\rangle$$

$$= |i f_{abc} X_c\rangle$$

$$= |[X_a, X_b]\rangle$$

Roots are weights of Adjoint Rep.

$$H_i |H_j\rangle = |[H_i, H_j]\rangle = 0$$

⇒ Cartan generators have zero weight vectors

Cartan states are orthonormal

$$\langle H_i | H_j \rangle = \lambda^{-1} \text{Tr}(H_i H_j)$$

$$= \lambda^{-1} \cdot \lambda \delta_{ij}$$

$$= \delta_{ij}$$

Other states have non-zero weight vectors

$$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle$$

$$= |\alpha_i E_\alpha\rangle$$

$$= |[H_i, E_\alpha]\rangle$$

$$\Rightarrow [H_i, E_\alpha] = \alpha_i E_\alpha$$

These non-zero weights uniquely specify the states in

the Adjoint representation $| \mu, \alpha; D \rangle_{\text{Adj.}} = | \mu, D \rangle_{\text{Adj.}}$

The E_α can NOT be Hermitian

$$[H_i, E_\alpha]^\dagger = \alpha_i E_\alpha^\dagger \quad (\alpha_i \in \mathbb{R})$$

$$[E_\alpha^\dagger, H_i] = \alpha_i E_\alpha^\dagger$$

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

$$\Rightarrow E_\alpha^\dagger = E_{-\alpha}$$

SU(2)

$$H = [H_i, E_{\pm\alpha}] = \pm \alpha_i E_{\pm\alpha} \quad \left([J_3, J_{\pm}] = \pm J_{\pm} \right)$$

States corresponding to different weights must be orthogonal
 (different eigenvalues for at least 1 of Cartan generators)

$$\langle E_{\alpha} | E_{\beta} \rangle = \lambda^{-1} \text{tr} (E_{\alpha}^{\dagger} E_{\beta}) = \delta_{\alpha\beta} = \frac{\pi}{i} \delta_{\alpha_i \beta_i}$$

$\alpha_i = \text{roots}$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} = \text{root vector}$$

$E_{\pm\alpha} = \text{Raising \& lowering operators}$

$$\begin{aligned} H_i E_{\pm\alpha} |\mu, D\rangle &= ([H_i, E_{\pm\alpha}] + E_{\pm\alpha} H_i) |\mu, D\rangle \\ &= (\pm \alpha_i E_{\pm\alpha} + \mu_i E_{\pm\alpha}) |\mu, D\rangle \\ &= (\mu_i \pm \alpha_i) E_{\pm\alpha} |\mu, D\rangle \end{aligned}$$

Consider the state $E_{\alpha} |E_{-\alpha}\rangle$

zero weight $H_i E_{\alpha} |E_{-\alpha}\rangle = (\alpha - \alpha) E_{\alpha} |E_{-\alpha}\rangle = \phi$

\Rightarrow linear combination of states corresponding to Cartan generators

$\Rightarrow [E_{\alpha}, E_{-\alpha}] = \text{linear combination of Cartan generators}$

$$E_{\alpha} |E_{\alpha}\rangle = \rho_i |H_i\rangle = |\rho_i H_i\rangle = |[E_{\alpha}, E_{-\alpha}]\rangle$$

$$\Rightarrow [E_{\alpha}, E_{-\alpha}] = \beta \cdot H$$

What is β ?

$$\rho_i |H_i\rangle = E_{\alpha} |E_{-\alpha}\rangle$$

$$\langle H_j | \rho_i |H_i\rangle = \langle H_j | E_{\alpha} |E_{-\alpha}\rangle$$

$$\delta_{ij} \beta_i = \lambda^{-1} \text{tr}(H_j^{\dagger} [E_{\alpha}, E_{-\alpha}])$$

$$= \lambda^{-1} \text{tr}(E_{-\alpha} [H_j, E_{\alpha}])$$

$$= \lambda^{-1} \alpha_j \text{tr}(E_{\alpha} E_{\alpha}) \quad (\text{tr}(E_{\alpha}^{\dagger} E_{\alpha}) = \lambda)$$

$$\beta_j = \alpha_j$$

$$[E_{\alpha}, E_{-\alpha}] = \alpha_i H_i$$

$$\left(\begin{array}{c} \text{SU}(2) \\ [J_+, J_-] = J_3 \end{array} \right)$$

SU(2)_s

For each non-zero pair of root vectors, $\pm \alpha$, there is an SU(2) sub-algebra of the group, with generators

$$E^{\pm} \equiv \frac{1}{|\alpha|} E_{\pm \alpha}$$

$$E_3 = \frac{1}{|\alpha|^2} \alpha \cdot H$$

$$\Rightarrow [E_3, E^{\pm}] = \pm E^{\pm}$$

$$[E^+, E^-] = E^3$$

$$E_3 |\mu, x, D\rangle = \frac{\alpha \cdot \mu}{\alpha^2} |\mu, x, D\rangle \quad \text{in some rep. } D$$

$$\Rightarrow 2 \cdot \frac{\alpha \cdot \mu}{\alpha^2} = \mathbb{Z} \quad (E_3 | \rangle = J_3 | \rangle)$$

$J_3 = \text{integer or half-integer}$

For some integer P ,

$$(E^+)^{P+1} |\mu, x, D\rangle = 0 \quad \text{raise out of space}$$

$J^+ | + \rangle = \emptyset$

$$E^3 (E^+)^P |\mu, x, D\rangle = \frac{\alpha \cdot (\mu + P\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + P = j \quad \text{max } E_3$$

$(E^+)^P |\mu, x, D\rangle$

Similarly

$$(E^-)^{Q+1} |\mu, x, D\rangle = \emptyset$$

$$E_3 (E^-)^Q |\mu, x, D\rangle = \left(\frac{\alpha \cdot \mu}{\alpha^2} + Q \right) (E^-)^Q |\mu, x, D\rangle$$
$$= -j (E^-)^Q |\mu, x, D\rangle$$

$$2 \frac{\alpha \cdot \mu}{\alpha^2} + (P - Q) = 0$$

$$\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2} (P - Q)$$

These relations lead to a geometric interpretation of all compact Lie Groups

Lets apply to roots ($\mu = \alpha, \beta$)

Take $SU(2)$ sub algebra (distinct roots)

$\mu = \beta$:

$$E_{\alpha} | \beta, x, D \rangle \Rightarrow \frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2} (p - q)$$

$$\mu = \alpha: E_{\beta} | \alpha, x, D \rangle \Rightarrow \frac{\beta \cdot \alpha}{\beta^2} = -\frac{1}{2} (p' - q')$$

$$\Rightarrow \cos^2 \theta_{\alpha\beta} = \frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} = \frac{1}{4} (p - q)(p' - q')$$

$$p - q \in \mathbb{Z}$$

$(p - q)(p' - q') \in$ positive integers

$$p' - q' \in \mathbb{Z}$$

$$0 \leq \cos^2 \theta_{\alpha\beta} \leq 1$$

Only 4 Interesting Possibilities

$(p - q)(p' - q')$	$\theta_{\alpha\beta}$
0	90° ($\pi/2$)
1	60° or 120°
2	45° or 135°
3	30° or 150°
4	not interesting

$$0^\circ \Rightarrow \alpha = \beta$$

but uniqueness makes trivial

$$180^\circ \Rightarrow \alpha = -\beta$$

always come in \pm pairs, trivial

How does this look in $SU(3)$?

Standard Representation

$$t_a = \lambda_a / 2, \quad \text{tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$\{t_1, t_2, t_3\}$ $SU(2)$ subgroup of $SU(3)$
(Isospin)

$\Rightarrow t_3 \in$ Cartan subalgebra

$t_8 \in$ Cartan subalgebra also diagonal

$$H_1 = t_3, \quad H_2 = t_8$$

What are weights of the representation?

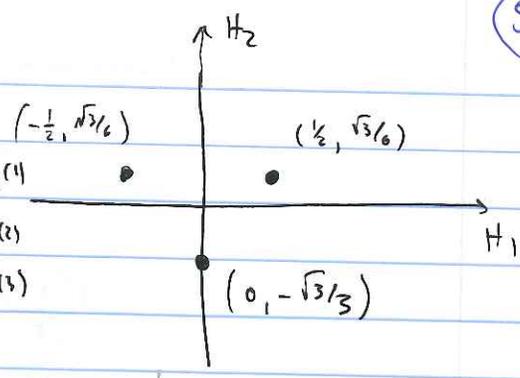
$$e\text{-vectors} \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$H_1 |1\rangle = \frac{1}{2} |1\rangle$$

$$H_2 |1\rangle = \sqrt{3}/6 |1\rangle$$

Weight vectors

- 1) $\rightarrow (\frac{1}{2}, \frac{\sqrt{3}}{6}) = \alpha^{(1)} \mu^{(1)}$
- 2) $\rightarrow (-\frac{1}{2}, \frac{\sqrt{3}}{6}) = \alpha^{(2)} \mu^{(2)}$
- 3) $\rightarrow (0, -\frac{\sqrt{3}}{3}) = \alpha^{(3)} \mu^{(3)}$



Now, find roots for non-zero weight vectors,
or find raising/lowering operators

$$[H_i, E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha}$$

recall, generators take us from one ^{weight} root to another

$$H_i \cdot E_{\pm\alpha} |\mu, D\rangle = (\mu_i \pm \alpha_i) E_{\pm\alpha} |\mu, D\rangle$$

so roots will be differences of weights

$$\begin{aligned} \mu^{(1)} - \mu^{(2)} &= (1, 0) \\ \mu^{(1)} - \mu^{(3)} &= (\frac{1}{2}, \frac{\sqrt{3}}{2}) \\ \mu^{(2)} - \mu^{(3)} &= (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \end{aligned}$$

We want generators such that

$$[H_1, E_{\pm}] = \pm 1 E_{\pm}, \quad [H_2, E_{\pm}] = \phi \cdot E_{\pm}$$

$$\Rightarrow E_{\pm, 0}$$

$$[H_1, E'_{\pm}] = \pm \frac{1}{2} E'_{\pm} \quad [H_2, E'_{\pm}] = \pm \frac{\sqrt{3}}{2} E'_{\pm}$$

$$\Rightarrow E_{\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2}}$$

$$\Rightarrow E_{\mp \frac{1}{2}, \pm \frac{\sqrt{3}}{2}}$$

from $SU(2)$, we know $\frac{J_1 \pm iJ_2}{\sqrt{2}}$ so how about

$$\frac{t_1 \pm it_2}{\sqrt{2}} = E_{\pm, 0}$$

by analogy $\frac{t_6 \pm it_7}{\sqrt{2}} = E_{\pm \frac{1}{2}, \frac{\sqrt{3}}{2}}$ $\left(\frac{t_4 \pm it_8}{\sqrt{2}} \right)$

$$\frac{t_4 \pm it_8}{\sqrt{2}} = E_{\pm \frac{1}{2}, \frac{\sqrt{3}}{2}}$$

$$\frac{t_1 + it_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[t_3, E_{\pm, 0}] = E_{\pm, 0}$$

$$\frac{t_1 - it_2}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[t_3, E_{-, 0}] = E_{-, 0}$$

$|1\rangle \leftrightarrow |2\rangle$

$$[t_8, E_{\pm, 0}] = 0 \cdot E_{\pm, 0}$$

$\downarrow \pm$

$$\frac{t_4 + it_8}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

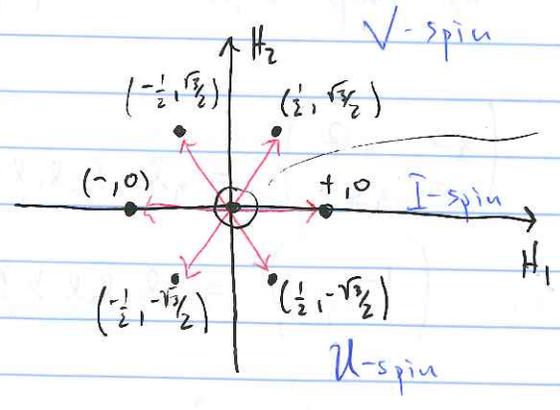
$$\frac{t_4 - it_8}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$|1\rangle \leftrightarrow |3\rangle$

$$\frac{t_6 \pm it_7}{\sqrt{2}}$$

$|2\rangle \leftrightarrow |3\rangle$ U_{\pm}

What do Roots look like?

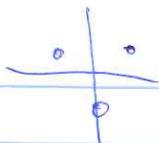


roots of t_3, t_8 $(0, 0)$

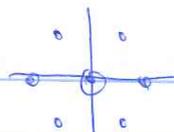
from experience

As we expect, these matrices have only one non-zero entry

These diagrams



fundamental



adjoint

represent two representations of $SU(3)$

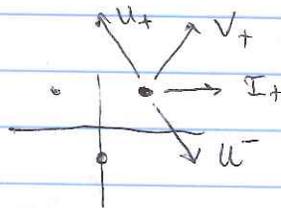
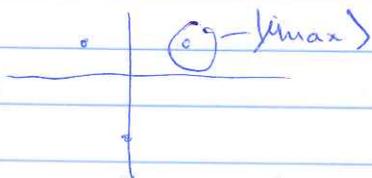
We can classify the states in the following way (non-unique but complete)

- find the maximum state $|I_{3max}\rangle$ such that with highest I_3 all raising operators annihilate state

$$I_+ |I_{3max}\rangle = 0 \quad U_+ |I_{3max}\rangle = 0, \quad V_+ |I_{3max}\rangle = 0, \quad U_- |I_{3max}\rangle = 0$$

e.g. $SU(3)$ $I_+ = \frac{t_1 + it_2}{\sqrt{2}}, \quad V_+ = \frac{t_4 + it_5}{\sqrt{2}}, \quad U_- = \frac{t_6 + it_7}{\sqrt{2}}$

In fundamental,



- Now, find max integer p , s.t.

$$(V_-)^p |I_{3max}\rangle \neq 0$$

Then find max integer q , s.t.

$$(I_-)^q (V_-)^p |I_{3max}\rangle \neq 0$$

} not unique way to count

- represent this state as

$$(p, q)$$

The total number of states in a representation

$$N_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

$$N_{(0,0)} = 1$$

$$N_{(1,0)} = 3$$

$$N_{(0,1)} = 3$$

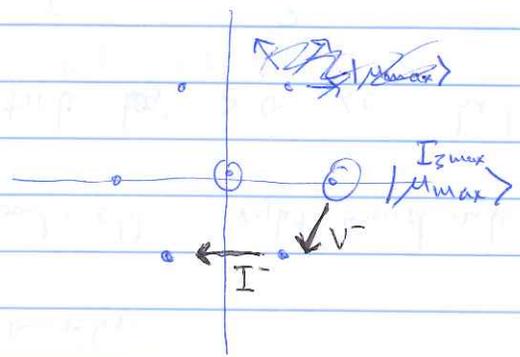
$$N_{(1,1)} = 8$$

$$N_{(2,0)} = 6$$

$$N_{(3,0)} = 10$$

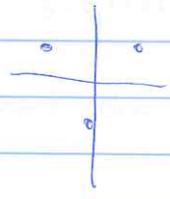
⋮

Notice $\oplus (1,1)$



What is $(0,1)$?

This is the conjugate state to $(1,0)$



The complex conjugate generators have the same algebra

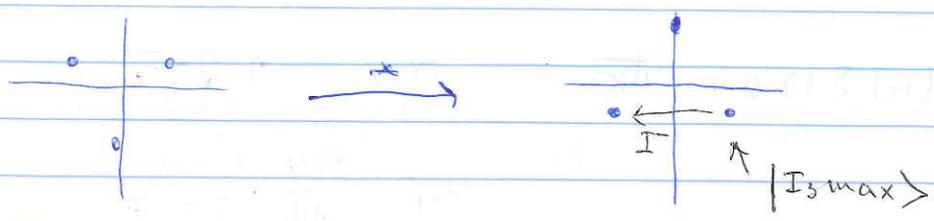
$$[t_a, t_b] = i f_{abc} t_c$$

$$[t_a, t_b]^* = -i f_{abc} t_c^*$$

$$[t_a^*, t_b^*] = -i f_{abc} t_c^*$$

$$[-t_a^*, -t_b^*] = +i f_{abc} (-t_c^*)$$

The conjugate weight diagram for SU(3)



~~quark~~

$$(I_-)^q (V_-)^p |I_{3max}\rangle$$

$$\Rightarrow (0, 1)$$

$$\text{quark} \equiv (1, 0)$$

$$\text{anti-quark} \equiv (0, 1)$$

$$(p, q)^* = (q, p) \text{ in general}$$

How can we combine these?

$$(1, 0) \otimes (0, 1) = ? \quad (1, 1) \oplus (0, 0)$$

To see this, let us return to Young Tableaux

before, I stated

$$18) \equiv \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array} = \frac{3 \cdot 4 \cdot 2}{3} = 8$$

Suppose we combine two 8's, what are the sub states

1) each horizontal row is at least as long as row below

2) horizontal = symmetrized
vertical = anti-symmetrized

⇒ $SU(n)$ has no more than n -rows

3) $SU(3)$: (p, q)

the first row has p more boxes than 2nd row

2nd row has q more boxes than 3rd row

3 □ = (1, 0)

6 □□ = (2, 0)

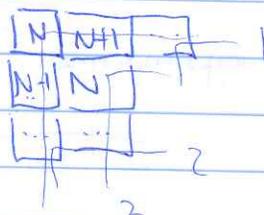
3 □□ = (0, 1)

8 □□ = (1, 1)

1 □□□ = (0, 0)

10 □□□ = (3, 0)

4) counting of states



$$\frac{\text{product of numbers}}{\text{product of hooks}}$$

OR
$$N_{(p,q)} = \frac{1}{2} (p+1)(q+1)(p+q+2)$$

5) Combining states

eg. $8 \otimes 8$

Combining States

$$8 \otimes 8 = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline a & a \\ \hline b & \\ \hline \end{array}$$

i) begin w/ left hand diagram

ii) add \boxed{a} 's in all ~~valid~~ ways to produce valid diagram

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} a a \oplus \begin{array}{|c|c|} \hline & \\ \hline a & \\ \hline & \\ \hline \end{array} a \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline a & \\ \hline & \\ \hline \end{array} a \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline a & \\ \hline \end{array}$$

Note only 1 a in a column

\boxed{a} 's are symmetric, so can not be anti-symmetric

iii) Starting in 2nd row, add b 's w/ constraint
of \boxed{a} 's must be \geq # \boxed{b} 's ~~in a row~~
from right to left

$$\begin{array}{|c|c|} \hline & \\ \hline b & \\ \hline & \\ \hline \end{array} a a \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline b & \\ \hline & \\ \hline \end{array} a a \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline a & b \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline a & \\ \hline b & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline a & \\ \hline b & \\ \hline \end{array} + \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline a & \\ \hline b & \\ \hline \end{array}$$

$$3 \times \bar{3}$$

$$3 \times 3$$

two quarks can "look" like $\bar{3}$